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SELF-SIMILAR SOLUTION OF THE PROBLEM OF GAS FLOW THROUGH A POROUS MEDIUM UNDER THE CONDITIONS OF TURBULENT FILTRATION*

A. G. BONDARENKO, V. M. KOLOBASHKIN and N. A. KUDRIASHOV

One-dimensional problem of isothermal gas flow through a porous medium under the conditions of turbulent filtration is considered. The gas filtration rate equation written in terms of the self-similar variables is reduced by changing the variables to an equation, a solution of which is represented by cylindrical functions. An analytical expression is obtained for the relation expressing the dependence of the gas pressure on the coordinates and time in the case when the gas enters the porous medium under constant pressure and the initial pressure within the medium is zero. Asymptotic approximations of the thermodynamic characteristics of the gas flow are analysed.

Let us consider the filtration of gas through a semi-infinite porous medium, with the gas pressure kept constant at the entry to the medium. A system of equations describing the motion has the form /1—6/

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{u})=0,\quad -\operatorname{grad}P=\frac{\mu}{k}\mathbf{u}+\frac{\rho}{b}|\mathbf{u}|\mathbf{u}\quad (P=c^2\rho) \tag{1}$$

Here P is pressure, ρ is density, $\mathbf{u}$ is the velocity of the filtering gas, μ is viscosity and k, b are the permeability coefficients for the laminar and turbulent filtration of the gas. The expression within the brackets holds in the case of an isothermal flow of perfect gas in a porous medium (c is the isothermal speed of sound).

Below we consider the case in which the linear velocity term in the Darcy law (second equation of (1)) can be neglected in comparison with the quadratic term. This can be done, as a rule, in the case of filtration through coarse-grained soils, and in high speed gas flows /2/.

For a plane flow under the quadratic filtration law, we obtain from (1)

$$\frac{\partial P}{\partial t}+\frac{\partial}{\partial x}(Pu)=0,\quad -\frac{\partial P}{\partial x}=\frac{Pu^2}{bc^2} \tag{2}$$

The self-similar solutions of the system (2) were studied in /4,5/. In /4/ the case of gas filtration with the gas flow specified at the boundary of the porous medium was considered. The integral curves of the equations in self-similar variables corresponding to the problem (2) were analysed and an asymptotic solution for the gas density obtained. In /5/ the case of constant pressure at the entry to the porous medium and constant initial pressure throughout the medium was studied, and an ordinary differential equation in self-similar variables was integrated numerically.

Below we obtain an exact analytic solution of the system (2) for the case of constant gas pressure at the boundary and zero intial gas pressure in the medium. We write the corresponding initial and boundary conditions for the system (2) in the form

$$P(x=0,t)=P_0,\quad P(x,t=0)=0 \tag{3}$$

We use the dimensionless quantity $\xi=x(bc^2t^2)^{-1/3}$ as the self-similar variable of the problem (2), (3). The pressure and velocity of the gas flow in the porous medium can be written, with the dimensionality taken into account, in the form

$$P=P_0f(\xi),\quad u=\left(\frac{bc^2}{t}\right)^{1/3}\varphi(\xi) \tag{4}$$

Substituting (4) into (2) and (3), we obtain the following system of ordinary differential equations and boundary conditions:

$$\frac{d(\varphi f)}{d\xi}=\xi\frac{df}{d\xi},\quad \varphi^2f+\frac{df}{d\xi}=0 \tag{5}$$

$$f(\xi=0)=1,\quad f(\xi\to\infty)=0 \tag{6}$$

Elimination of f from (5) yields the following equation for φ

$$d\varphi/d\xi+\xi\varphi^2-\varphi^3=0 \tag{7}$$

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Performing in (7) the substitution

$$\varphi = (^1/_2\,\xi^2 - \psi)^{-1} \tag{8}$$

where ψ is an independent and ξ a dependent variable, we obtain for $\xi(\psi)$ the Ricatti equation in the form

$$d\xi / d\psi + \psi - {}^1/_2\xi^2 = 0$$

Setting

$$\xi = -2y' / y \tag{9}$$

we obtain an equation for $y(\psi)$

$$y'' - {}^1/_2\psi y = 0$$

the solution of which is represented by the Airy functions /7/

$$y = \sqrt{\psi}\left[C_1 I_{1/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right) + C_2 K_{1/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right)\right], \quad \psi > 0, \quad y = \sqrt{|\psi|}\left[C_3 J_{1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right) + C_4 J_{-1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right)\right], \quad \psi < 0 \tag{10}$$

Here $J_{1/3}(x)$, $J_{-1/3}(x)$ is a Bessel function of a real argument, $I_{1/3}(x)$ is a Bessel function of an imaginary argument, and $K_{1/3}(x)$ is a Macdonald function.

Let us obtain an expression for the gas pressure within the porous medium. The second equation of (5) with the formulas given below taken into account, yields

$$f = \exp\{-\textstyle\int \varphi^2 d\xi\} = \varphi^{-1} y^2$$

The above expression was obtained with help of the following formulas:

$$\int \varphi^2 d\xi = \ln\varphi + \int \xi\varphi d\xi, \quad \int \xi\varphi d\xi = \int \xi d\psi = -\ln y^2$$

From this we obtain the following expression for the gas pressure within the porous medium:

$$f = \psi\varphi^{-1}\left[C_1 I_{1/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right) + C_2 K_{1/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right)\right]^2, \quad \psi > 0, \quad f = \psi\varphi^{-1}\left[C_3 J_{1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right) + C_4 J_{-1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right)\right]^2, \quad \psi < 0 \tag{11}$$

C_1, C_2, C_3 and C_4 in (10) and (11) are constants determined from the conditions (6). Using the recurrence relations for cylindrical functions /8/ we obtain, from (9), the following transcendental equation for $\psi = \psi(\xi)$:

$$\left[C_1 I_{-2/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right) - C_2 K_{2/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right)\right]\left[C_1 I_{1/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right) + C_2 K_{1/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right)\right]^{-1} = -\frac{\xi}{\sqrt{2\psi}}, \quad \psi > 0 \tag{12}$$

$$\left[C_3 J_{-2/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right) - C_4 J_{2/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right)\right]\left[C_3 J_{1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right) + C_4 J_{-1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right)\right]^{-1} = \frac{\xi}{\sqrt{2|\psi|}}, \quad \psi < 0$$

Equations (8), (11) and (12) represent formal solutions of the system of equations describing a gas flow with convective acceleration, through a porous medium.

Let us inspect the character of the dependence of ψ on ξ. When $\xi = 0$, (8) yields $\psi(\xi = 0) = -\varphi_0^{-1}$. Since the velocity of motion $\varphi > 0$, the gas pressure in the porous medium is described, at small values of ξ, by the second equation of (11).

We shall show that $\psi \approx {}^1/_2\,\xi^2$ when $\xi \gg 1$. From (7) we obtain the asymptotic expression for the velocity of the filtering gas at $\xi \gg 1$

$$\varphi \simeq \xi + \xi^{-2} + o(\xi^{-2}) \tag{13}$$

Using (8), we obtain

$$\psi \simeq {}^1/_2\xi^2 - \xi^{-1} + o(\xi^{-1}) \tag{14}$$

Figure 1 shows, in solid line, the approximate dependence of ψ on ξ. The dash-dot line depicts the function $\psi_1(\xi) = {}^1/_2\xi^2$, the latter being an asymptotic for $\psi(\xi)$. Other relations connecting $\psi(\xi)$ with the asymptotic (14) are in disagreement with the physical sense of the problem: the gas velocity in a porous medium has no negative value (when $\xi \in [0, \infty)$ $d\psi / d\xi = \varphi(\xi) > 0$).

Since $\psi \to \infty$ as $\xi \to \infty$, we must set C_1 equal to zero in (10) — (12) so that the gas pressure in the porous medium does not become infinite when $\xi \to \infty$. From (12) we obtain the following transcendental equation for determining $\psi(\xi)$ at $\psi > 0$

$$\sqrt{2\psi}\, K_{2/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right) = \xi K_{1/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right) \tag{15}$$

Using the asymptotic behavior of the Macdonald function /9/ as $\psi \to 0$ we find, from (15), ξ_0

for which $\psi(\xi_0) = 0$

$$\xi_0 = \sqrt{2}\,(3\sqrt{2})^{1/3}\,\Gamma\left(\frac{2}{3}\right)\Gamma^{-1}\left(\frac{1}{3}\right) \approx 0.37 \tag{16}$$

Next we determine the constants C_2, C_3 and C_4 in (10)–(12). The gas pressure in the porous medium is continuous when $\psi = 0$, therefore equating (11) we obtain the relation

$$C_2 = \sqrt{3}C_4 / \pi$$

Passing in (12) to the limit as $\psi \to 0$ and taking (16) into account, we find that $C_3 = C_4$.

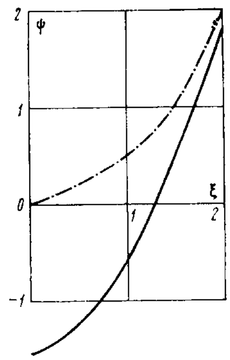

Fig.1

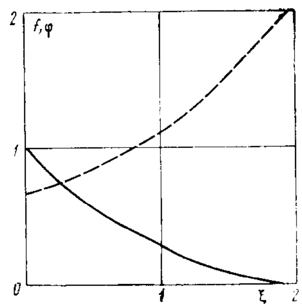

Fig.2

The transcendental equation for calculating the dependence of ψ on ξ for $\psi < 0$ is obtained from (12), and has the form

$$\left[J_{-2/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right) - J_{2/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right)\right]\left[J_{1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right) + J_{-1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right)\right]^{-1} = \frac{\xi}{\sqrt{2|\psi|}} \tag{17}$$

The value of $\psi(\xi = 0)$ is found from (17) for $\xi = 0$, and is equal to $\psi(0) = -1.52$, while the expression (8) yields $\varphi(\xi = 0) = 0.65$.

Substituting $\psi(0)$ and $\varphi(0)$ into (11) and remembering that $f(\xi = 0) = 1$, we obtain $C_3 = 1.2$. The relations connecting the constants now yield $C_4 = 1.2$, $C_2 = 0.65$.

For the conditions (6) the expression for the gas pressure in a porous medium has the form

$$f = 1.2\psi\varphi^{-1}\left[J_{1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right) + J_{-1/3}\left(\frac{\sqrt{2}}{3}|\psi|^{3/2}\right)\right]^2, \quad \psi \leqslant 0 \quad (\xi \leqslant \xi_0), \quad f = 0.65\psi\varphi^{-1}K_{1/3}\left(\frac{\sqrt{2}}{3}\psi^{3/2}\right), \quad \psi \geqslant 0 \quad (\xi \geqslant \xi_0) \tag{18}$$

Figure 2 shows that solution of the system of equations (5) with conditions (6). The dashed line corresponds to the gas filtration velocity $\varphi(\xi)$ and the solid line represents the gas pressure in the porous medium. Taking into account (13) and (14) we obtain, from (18), the expression describing the asymptotic behavior of the gas pressure when $\xi \gg 1$

$$f \approx 6\xi^{-2}\exp\{-\xi^3/3\}$$

In conclusion, we pause to discuss the correspondence between the system of equations (1), (2) with the boundary and initial conditions (6), and a real gas flow through a porous medium. A rigorous solution of the problem (for all x and t) should be based on a system of equations, in which the equations of motion (2) are replaced by the Euler equation taking the frictional forces into account. Estimates show that for small x $(x \ll b)$ and t $(t \ll kP_0\mu^{-1}c^{-2})$ the total, with respect to time derivative of velocity should not be neglected since this restricts the range of the values of x and t for which the system (2) describes the gas flow in a porous medium. Equation (2) ceases to hold at small x and t, nevertheless, numerical computations of the system of gasdynamic equations with and without taking into account the inertial terms, have shown that the solutions are practically indistinguishable from each other within the range of values $x \gg b$ and $t \gg kP_0\mu^{-1}c^{-2}$. This implies that, within the range of values $x \gg b$ and $t \gg kP_0\mu^{-1}c^{-2}$ the system of equations (2) with conditions (3) describes, with a fair degree of accuracy, the motion of gas in a porous medium.

The results of the present paper can be used not only in the analysis of convective filtration of gas, but also in studying gas flows along a plane crack at high flow rates /6/.

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